NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.

IMM-NYU 165

IMM/165

ON THE NON-OCCURRENCE OF A LIMITING LINE IN TRANSONIC FLOW

By

K. O. FRIEDRICHS

Prepared under
Navy Contract N6ori-201 Task Order No. 1
By the
Institute for Mathematics and Mechanics
New York University

December, 1947

IMM-165
c.1

On the Non-Occurrence of a Limiting Line in Transonic Flow

by

K. O. Friedrichs

If the free stream Mach number of the flow of air or any compressible fluid past an airfoil is sufficiently small the flow is purely subsonic. If the Mach number exceeds a certain critical value, a region of supersonic flow will appear adjacent to a section of the airfoil. If the Mach number is further increased, shock fronts appear in the flow according to experimental evidence. It was natural to conjecture that this breakdown of the continuous flow is connected with the occurrence of a limiting line in the mathematical representation of the flow. We shall prove that this is not the case and that, therefore, the breakdown must have other causes.

It was natural to hold the limiting line responsible for the breakdown because for certain other types of flow the limiting line is, in a sense, responsible for the appearance of shock fronts. Consider, for example, a supersonic flow along a wall which is first plane and is then curved inward toward the flow. The flow is parallel and constant up to the Mach line issuing from the point at which the wall starts to bend. Beyond this Mach line the flow passes through a simple wave, a compressive Prandtl-Meyer wave, see [1] or [2]. It is not possible that the flow consists just of this simple wave; for, the straight Mach lines of this wave form an envelope, a limiting line. Instead, a shock front develops from the point at

NEW YORK UNIVERSITY
INSTITU[illegible] CES

which the cusp of the limiting line would have been. Thus the flow avoids reaching a limiting line (except at the cusp).

In flows which are not composed just of simple waves, limiting lines have been found by several authors. In some special cases they were observed by G. I. Taylor [3], more generally by F. and M. Clauser [4], Tollmien [5], Ringleb [6]; for a general discussion of its significance see v. Kármán [7] and Tsien [8].

The limiting line occurs, of course, only in the mathematical representation of a flow by the hodograph method. Let u and v be the velocity components of the two-dimensional flow which takes place in the (x,y)-plane. If the pair (x,y) can be considered a function of u and v, it satisfies a certain linear differential equation. A solution (x,y) of this differential equation represents a flow only if the Jacobian $J = y_u x_v - x_u y_v$ does not vanish. If J = 0 along a certain curve L in the (u,v)-plane, the mapping of the (u,v)-plane into the (x,y)-plane has a fold. The edge of this fold, the image of the curve L, is the limiting line. (For a detailed discussion see [8] and [2].) If a solution (x,y) for which such a limiting line occurs is the only one under certain conditions, then under these conditions no continuous flow exists.

We now ask for flows past an airfoil or flows through a duct with a constriction, assuming that the Mach number M of the free stream, or of the entrance and exit flow in the duct, is less than one, but that the flow is supersonic in some limited region adjacent to the profile or to the wall of the duct near the constriction. We assume that we have analytic functions (x,y) of (u,v) satisfying the appropriate differential equations and the boundary condition that the image of the boundary of the domain in the (u,v)-plane in which the pair (x,y) is defined lies on the prescribed contour in the (x,y)-plane, of which we require that it has everywhere a finite curvature. We assume that we have a set of such functions depending continuously on the Mach number M, such that the Jacobian J does not vanish for $M < M_0$, while $J = 0$ somewhere for $M \geq M_0$. Our statement is that that is not possible.*

In Section 3 we shall dispose of the possibility that the limit line might first appear at the boundary; in other words, that $J = 0$ at some point of the boundary for $M = M_0$. It is known that the curvature of a streamline becomes infinite on approaching a single limiting line. Since the prescribed contour is a streamline but has a finite curvature, a single limiting line can never appear at the boundary. It is possible, however, that the curvature of the streamline remains finite on approaching a point at which two limiting lines cross. Nevertheless, as will be shown in Section 3, it remains true under the circumstances here considered that it is a consequence of the finiteness of the wall curvature that the Jacobian does not vanish at the wall.

We shall dispose of the possibility that the limiting line might first appear in the interior of the flow region. Since $J \neq 0$ for $M < M_0$ and $J \geq 0$ in the region of subsonic flow, we have $J > 0$ for $M < M_0$ and hence $J \geq 0$ for $M = M_0$. In Section 4 we shall consider the case that $J = 0$ for $M = M_0$ at some point 0 in the region of supersonic flow.

We shall prove that the condition that $J \geq 0$ in the neighborhood of 0 implies successively that the derivatives of x and y with respect to u and v of all orders vanish. Since x and y are assumed analytic, it follows that x and y are constant and hence do not represent a flow.

---

* Similar results seem to have been obtained by Nikolsky and Taganov [11]; the author has not yet had the opportunity of seeing this paper. A discussion of singularities resulting from small disturbances in transonic flow was given by Guderley [12].

Finally we should dispose of the case that $J = 0$ for $M = M_0$ at some point at which the flow speed is sonic. This case, which will be treated in a later publication, causes greater difficulties. The main difficulty lies in a certain intricate algebraic problem, which was solved by D. A. Flanders.

Once it has been proved that no limiting line can occur in a transonic flow of the indicated type, one wonders what causes the breakdown of the flow when the Mach number reaches a certain critical value. Of course, one does not even know for certain whether or not such flow solutions exist at all for arbitrary contours. If these solutions exist for $M < M_0$, then on approaching M_0 a different kind of singularity must occur. It is inconceivable that flow solutions should exist for all Mach numbers $M < 1$. A strong indication in this direction is the fact, proved by Görtler [9], that in a symmetrical duct the two separated regions of supersonic flow can never join. Also, the one-dimensional hydraulic treatment of nozzle flow indicates strongly that a symmetrical flow through a duct ceases to exist when the average speed in the narrowest cross section has reached sound speed. The type of breakdown that occurs in the flow through a symmetrical duct may also be expected to occur in flow past an airfoil.

Whether or not such a breakdown of the existence of a continuous flow is responsible for the appearance of shock fronts in real flows is doubtful. There are strong indications that the viscous boundary layer along the contour is decisive in the formation of shocks. The experimental evidence, gathered and interpreted by Liepmann [10], shows that the distortion of the flow due to the boundary layer is such as to produce a compression wave in which a limiting line would occur if a shock did not interfere.

1. Formulation of the problem.

The two-dimensional gas flow will be described by giving the velocity components $u = q\cos\theta$ and $v = q\sin\theta$ as functions of the coordinates x and y. These functions satisfy the differential equations

$$(1.01)\quad (c^2-u^2)u_x - uv(u_y+v_x) + (c^2-v^2)v_y = 0, \qquad u_y-v_x = 0,$$

in which the sound speed c is a given function of the flow speed q having the property that q^2-c^2 is an increasing function of q.

We consider flows in a duct possessing a constriction having a subsonic flow speed at entrance and exit, and a supersonic flow speed in some region in the middle portion of the duct. We also assume that nowhere cavitation takes place, i.e. that $c > 0$ throughout. The walls of the duct are supposed to be smooth curves with continuously turning tangents and finite curvature, at least where the region of supersonic or sonic flow is adjacent to it.

The regions of subsonic, sonic, or supersonic flow will be denoted by R, S, T respectively.

We assume that the flows possess simple hodographs in the (u,v)-plane so that they can be represented by functions $x(u,v)$, $y(u,v)$ defined in some domain of the (u,v)-plane, satisfying the differential equations

$$(1.02)\quad (c^2-u^2)y_v + uv(y_u+x_v) + (c^2-v^2)x_u = 0, \qquad y_u-x_v = 0.$$

We now consider a set of such flows through the same duct depending on a parameter, t, which may, e.g., represent the Mach number of the incoming flow; we might also let the shape of the duct walls depend on a parameter, keeping the entrance Mach number fixed. Thus we have a set of solutions of the differential equations (1.02) which depend on a parameter t and which are defined in a domain in the (u,v)-plane which also depends on the parameter. It is essential for the following argument that the solutions $x(u,v)$, $y(u,v)$ are non-constant analytic functions of u and v while the dependence of x,y and the derivatives x_u, x_v, y_u, v_y on the parameter t is only required to be continuous.

These functions $x(u,v)$, $y(u,v)$ are defined in some region D in the (u,v)-plane, which also depends on the parameter t. The boundary B of this region is mapped into the prescribed walls. We assume that also the boundary B depends continuously on t (in the sense e.g. that B can be given in a parametric representation which depends continuously on t).

We now assume that for each value of the parameter t less than a certain critical value t_o the solution $x(u,v)$, $y(u,v)$ represents a flow through the prescribed duct with subsonic speed at entrance and exit, while for values of $t > t_o$ this is no longer the case owing to the occurrence of a limiting line. Our statement is that this is impossible. In other words, *if solutions* $x(u,v)$, $y(u,v)$ *fail to lead to a flow through the duct when the parameter passes through a critical value, the reason is not the appearance of a limiting line*.

If a limiting line occurs the Jacobian

$$J = y_u x_v - y_v x_u \tag{1.03}$$

of the transformation of the (u,v)-plane into the (x,y)-plane vanishes somewhere in the regions T or S of supersonic or sonic flow.

As is well known the Jacobian is non-negative for subsonic flow. (This follows immediately from equations (1.02): introducing coordinates at the point in question such that $u = q$, $v = 0$ there, equation (1.02) reduces to $(c^2-q^2)y_v + c^2x_u = 0$, whence $J = y_u^2 + (M^2-1)y_v^2 \geq 0$.)

In accordance with these facts we specify our assumption: We assume that for $t < t_o$ the Jacobian J is non-negative throughout and in particular positive in the regions T and S of supersonic or sonic flow. For $t = t_o$ we assume that $J = 0$ at some point in T or S. (We also exclude the occurrence of cavitation, i.e. we require $c > 0$ throughout for $t = t_o$.) Clearly, J is nowhere negative for $t = t_o$; for then J would be negative somewhere for $t < t_o$ contrary to assumption.

We shall first exclude the possibility that $J = 0$ at the wall. Once this is done we need only prove the

Theorem: If for an analytic solution x,y of the equations (1.02) the Jacobian J vanishes at some point in the interior of the domain of definition at which $q \geq c$, then J is negative somewhere.

2. Appropriate quantities.

It is convenient to use the polar coordinates q,θ and introduce the quantities

$$(2.01) \qquad X = x\cos\theta + y\sin\theta, \qquad Y = y\cos\theta - x\sin\theta$$

instead of x and y. We further introduce the quantity

$$(2.02) \qquad s = s(q) = M^2 - 1 = \frac{q^2-c^2}{c^2} = \cot^2\alpha,$$

M being the Mach number, α the Mach angle. Equations (1.02) then take the form

$$(2.03) \qquad qX_q = s(Y_\theta+X), \qquad qY_q = X_\theta-Y.$$

From the relation

$$Y_qX_\theta - Y_\theta X_q = y_qx_\theta - y_\theta x_q + Y_qY + X_qX$$

we eliminate X_q and X_θ by (2.03) and observe

(2.04) $$y_qx_\theta - y_\theta x_q = qJ.$$

Then we obtain

(2.05) $$J = Y_q^2 - sq^{-2}(Y_\theta+X)^2.$$

In the regions T and S, where $s \geq 0$ by (2.02), relation (2.05) leads to

(2.06) $$J \leq Y_q^2.$$

3. The limit line does not appear at the wall.

To eliminate the possibility that $J = 0$ somewhere at the walls we express the curvature of any streamline in terms of the Jacobian.

We first consider the flow for $t < t_0$. Then the Jacobian is positive at least in the regions T and S, and x and y can be introduced as independent variables. Since θ is the angle of inclination of the streamline against the x-axis, the curvature along any streamline is given by

(3.01) $$\kappa = \theta_x \cos\theta + \theta_y \sin\theta.$$

In view of (2.04) we have the inversion formulas

(3.02) $$Jq\theta_x = y_q, \quad Jq\theta_y = -x_q;$$

inserting them into (3.01) we find

$$(3.03) \qquad Jq\kappa = Y_q,$$

whence by (2.06)

$$(3.04) \qquad J^2q^2\kappa^2 \geq J,$$

or, since $J > 0$ in T+S for $t < t_o$,

$$(3.05) \qquad \kappa^2q^2J \geq 1.$$

Since each wall is a streamline, relation (3.05) holds along the boundary curve B in the (u,v)-plane which is mapped into the walls of the duct. Let κ_m be the maximum of the absolute curvature $|\kappa|$ along the walls and let $\hat{q}$ be the limit speed. Then (3.05) yields the inequality

$$(3.06) \qquad J \geq 1/\kappa_m^2\,\hat{q}^2 \quad \text{along B.}$$

This relation holds for $t < t_o$, but the right-hand side is independent of t. Since it was assumed that x_u, x_v, y_u, y_v and hence $J(u,v)$ depend continuously on t, and also that B depends continuously on t, it follows that inequality (3.06) also holds for $t = t_o$. Thus $J > 0$ along the walls for $t = t_o$.

4. The limit line does not appear in the supersonic region.

In order to prove the theorem formulated at the end of Section 1 for a point in the region T of supersonic flow, it is convenient to transform the differential equations (1.02) into characteristic form:

$$(4.01) \qquad y_\sigma = \tan(\alpha+\theta)x_\sigma, \qquad y_\tau = -\tan(\alpha-\theta)x_\tau.$$

Without restriction we may assume that $|\theta| < \alpha$ and

$|\theta| < \frac{\pi}{2} = \alpha$ in a neighborhood N of the point considered. The characteristic parameters σ and τ are to be analytic functions of q and θ satisfying the relations

$$u_\sigma = \tan(\alpha-\theta)v_\sigma, \qquad u_\tau = -\tan(\alpha+\theta)v_\tau$$

or

$$(4.02) \qquad q_\sigma = q \tan\alpha\,\theta_\sigma, \qquad q_\tau = -q \tan\alpha\,\theta_\tau,$$

and so chosen that none of the derivatives $q_\sigma, \theta_\sigma, q_\tau, \theta_\tau$ vanishes in the neighborhood of the point considered; this is possible since $\tan\alpha \neq 0, \neq \infty$ for $0 < c < q$. (Note that $c > 0$ was assumed.)

We eliminate the quantity y from equations (4.01) and obtain after some obvious computation

$$(4.03) \qquad x_{\sigma\tau} = ax_\sigma + bx_\tau$$

with

$$(4.04) \qquad \begin{aligned} a &= -\frac{\sec(\alpha+\theta)\cos(\alpha-\theta)}{\sin 2\alpha}(\alpha+\theta)_\tau, \\ b &= -\frac{\sec(\alpha-\theta)\cos(\alpha+\theta)}{\sin 2\alpha}(\alpha-\theta)_\sigma. \end{aligned}$$

We maintain that the quantities $(\alpha+\theta)_\tau$ and $(\alpha-\theta)_\sigma$ do not vanish. For by (4.02.1) we have

$$(4.05.1) \qquad -q \sin\alpha(\alpha-\theta)_\sigma = q(\cos\alpha)_\sigma + \cos\alpha\, q_\sigma = (q \cos\alpha)_\sigma$$

$$= (\sqrt{q^2-c^2})_\sigma \ ;$$

similarly, we have

$$(4.05.2) \qquad -q \sin\alpha(\alpha+\theta)_\tau = (\sqrt{q^2-c^2})_\tau .$$

Since by assumption q^2-c^2 is an increasing function of q, and $q_\sigma \neq 0$, $q_\tau \neq 0$, our statement follows.

Since by assumption $|\theta| < \alpha$ and $|\theta| < 90^\circ - \alpha$ in the neighborhood N, we have

$$(4.06) \qquad a \neq 0, \quad b \neq 0 \quad \text{in } N.$$

Aside from a factor, the Jacobian J reduces to

$$(4.07) \qquad y_\sigma x_\tau - y_\tau x_\sigma = [\tan(\alpha+\theta) + \tan(\alpha-\theta)]x_\sigma x_\tau$$

$$= \frac{\sin 2\alpha}{\cos(\alpha+\theta)\cos(\alpha-\theta)} x_\sigma x_\tau,$$

as seen from (4.01). Since $\cos(\alpha+\theta)\cos(\alpha-\theta) > 0$ in N, we see that the sign of the Jacobian is that of the product $x_\sigma x_\tau$ in N.

We now assume that $J = 0$ at a certain point $Q = \{\sigma^0, \tau^0\}$ in the interior of D with $q^0 > c^0$. We want to show that $J < 0$ somewhere in the neighborhood of Q.

Without restriction we may assume $\sigma^0 = \tau^0 = 0$ and also $\theta^0 = 0$. We then set

$$(4.08) \qquad t^0 = \tan\alpha^0,$$

knowing that $t^0 > 0$. According to (4.07) we have $x_\sigma^0 x_\tau^0 = 0$. We assume that $x_\sigma x_\tau \geq 0$ in N and shall show that then x is constant. We shall do this by proving that the derivatives of x of every order vanish at 0.

Since $x_\sigma^0 x_\tau^0 = 0$, either $x_\sigma^0 = 0$ or $x_\tau^0 = 0$. Suppose we have

$$(4.09) \qquad x_\sigma^0 = 0.$$

Then we conclude from (4.03)

(4.10) $$x^0_{\sigma\tau} = b^0 x^0_\tau .$$

We now expand the product $x_\tau x_\sigma$ with respect to powers of τ, setting $\sigma = 0$. From (4.09) we see that this expansion begins with

$$x_\tau x_\sigma = x^0_\tau x^0_{\sigma\tau}\tau + \cdots .$$

Since $x_\tau x_\sigma \geq 0$ in N, the same inequality holds for the first term in the expansion of $x_\tau x_\sigma$; hence for all values of τ,

$$x^0_\tau x^0_{\sigma\tau}\tau \geq 0,$$

whence we conclude $x^0_\tau x^0_{\sigma\tau} = 0$.

This relation combined with (4.10) yields $x^0_\tau = 0$ since $b^0 \neq 0$ by (4.06). Thus, in view of (4.09) we have proved

(4.11) $$x^0_\sigma = x^0_\tau = 0.$$

We now proceed by induction. Suppose we have proved, for $n \geq 1$,

(4.12) $$x^0_\sigma = x^0_\tau = \cdots = x^0_{\sigma^n} = x^0_{\sigma^{n-1}\tau} = \cdots = x^0_{\sigma\tau^{n-1}} = x^0_{\tau^n}.$$

By differentiation of equation (4.03) we then derive

(4.13) $$x^0_{\sigma^n\tau} = \cdots = x^0_{\sigma\tau^n} = 0,$$

(4.14) $$x^0_{\sigma^n\tau^2} = \cdots = x^0_{\sigma^2\tau^n} = 0 \quad \text{for } n > 1,$$

(4.15) $$x^0_{\sigma^{n+1}\tau} = a^0 x^0_{\sigma^{n+1}}, \quad x^0_{\sigma\tau^{n+1}} = b^0 x^0_{\tau^{n+1}}.$$

Using relations (4.13) and (4.14) we see that the expansions of x_σ and x_τ with respect to powers of τ, taken for $\sigma = 0$, begin with

$$x_\sigma = \frac{1}{\underline{|n+1}}\, x^{o}_{\sigma\tau^{n+1}}\, \tau^{n+1} + \cdots ,$$

$$x_\tau = \frac{1}{\underline{|n}}\, x^{o}_{\tau^{n+1}}\, \tau^{n} + \cdots .$$

For $\sigma = 0$ we therefore have

$$x_\tau x_\sigma = \frac{1}{\underline{|n}}\, \frac{1}{\underline{|n+1}}\, x^{o}_{\tau^{n+1}} x^{o}_{\sigma\tau^{n+1}}\, \tau^{2n+1} + \cdots .$$

From $x_\tau x_\sigma \geq 0$ we then infer

$$x^{o}_{\tau^{n+1}} x^{o}_{\sigma\tau^{n+1}}\, \tau^{2n+1} \geq 0,$$

whence we conclude

$$x^{o}_{\tau^{n+1}} x^{o}_{\sigma\tau^{n+1}} = 0.$$

From (4.15) and $b^{o} \neq 0$ we then deduce

$$x^{o}_{\tau^{n+1}} = 0. \tag{4.16}$$

Interchanging σ and τ in the argument, we find

$$x^{o}_{\sigma^{n+1}} = 0. \tag{4.17}$$

Relations (4.13), (4.16), and (4.17) now give relations (4.12) for n+1 instead of n. The induction proof is thus completed.

It follows that all derivatives of x vanish at the point 0; hence the analytic function x is constant in contradiction to the assumption.

Thus we have proved that a limiting line cannot appear in a region in which the flow is supersonic. Earlier, in Section 3, we had shown that it cannot occur at the wall and in a later publication we shall prove that it cannot appear at a point at which the flow is sonic.

Bibliography

[1] Supersonic Flow and Shock Waves, AMP Report 38.2R, August 1944.

[2] Courant, R., and Friedrichs, K. O.: Supersonic Flow and Shock Waves, to appear.

[3] Taylor, G. I.: Recent work on the flow of compressible fluids. Journal of the London Mathematical Society, Volume 5, p. 224, 1930.

[4] Clauser, F. and M.: New Methods of Solving the Equations for the Flow of a Compressible Fluid, Ph. D. Thesis, California Institute of Technology (unpublished), 1937.

[5] Tollmien, W.: Zum Uebergang von Untershall--in Ueberschallströmungen, Zeitschrift für Angewandte Mathematik und Mechanik, Volume 17, pp. 117-136, 1937.

[6] Ringleb, F.: Exakte Lösungen der Differential gleichungen einer adiabatischen Gasströmung, Zeitschrift für Angewandte Mathematik und Mechanik, Volume 20, pp. 185-198, 1940.

[7] Kármán, Th. v.: Compressibility Effects in Aerodynamics, Journal of the Aeronautical Sciences, Volume 8, pp. 337-356, 1941.

[8] Tsien, H. S.: The Limiting Line in Mixed Subsonic and Supersonic Flow of Compressible Fluids, Technical Note No. 961, National Advisory Committee for Aeronautics, 1944.

[9] Görtler, H.: Zum Uebergang von Unterschall--zu Ueberschallgeschwindigkeiten in Düsen, Zeitschrift für Angewandte Mathematik und Mechanik, Volume 19, pp. 325-337, 1939.

[10] Liepmann, H. W.: Investigations of the Interaction of Boundary Layer and Shock Waves in Transonic Flow, Guggenheim Aeronautical Laboratory, California Institute of Technology.

[11] Nikolsky, A. A., and Taganov, G. I.: Gas Motion in a Local Supersonic Region and Conditions of Break of Potential Flow, Akademiya Nauk, S.S.S.R. Prikladnaya Matematika i Mechanika, Volume 10, pp. 481-502, 1946.

[12] Guderley, G.: The Reason for the Appearance of Compression Shocks in Transonic Flow (Monograph Transonic Flow), a translation by Kate Liepmann, Guggenheim Aeronautical Laboratory, California Institute of Technology.

IMM-NYU 165

AUTHOR

Friedrichs, K. O.

TITLE

On the non-occurrence...

NEW YORK UNIVERSITY
INSTITUTE [illegible]
25 Waverly Place, New York 3, N. Y.

Date Due

[illegible]			

[illegible]

[illegible]